THERMAL CONVECTION WAVES IN VISCOELASTIC FLUIDS

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We demonstrate that undamped and weakly damped thermal convection waves, in particular, thermal and transverse waves, can be propagated in viscoelastic heat-conducting fluids. We have found the frequency spectrum and the wavelengths for such waves. We indicate the possibility of developing a mechanical generator of thermal oscillations.

Let us examine the propagation of small perturbations in temperature, velocity, and density in a viscoelastic heat-conducting compressible fluid that is exclusively thermal. We will demonstrate that in the presence of a gravitational field, as well as in the presence of a special temperature field, these perturbations may be propagated in the form of undamped and weakly damped thermal convection waves.

1. There are several mathematical models of viscoelastic fluids [1-4]. We will use the isotropic Maxwell model with a single relaxation time τ [1-4]:

$$\tau \frac{\partial \sigma_{kj}}{\partial t} + \sigma_{kj} = \mu \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k} - \frac{2}{3} \delta_{kj} \frac{\partial v_n}{\partial x_n} \right) + \eta \delta_{kj} \frac{\partial v_n}{\partial x_n}, \tag{1}$$

where σ_{kj} denotes the components of the viscous stress tensor; v_k denotes the velocity components; and μ and η are viscosity coefficients. We will refer to the viscoelastic medium subject to the rheological equation of state (1) as a Maxwell fluid, which is the practice in the literature. The nonsteady motion and thermal processes in Maxwell fluids in a gravitational field are described by the following system of equations:

$$\rho \left(\frac{\partial v_k}{\partial t} + v_j \frac{\partial v_k}{\partial x_j} \right) = \frac{\partial P_{kj}}{\partial x_j} + \rho g_k, \quad P_{kj} = -\delta_{kj} p + \sigma_{kj}, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} (\rho v_k) = 0, \tag{3}$$

$$\rho c \left(\frac{\partial T}{\partial t} + v_k \frac{\partial T}{\partial x_k} \right) = -\frac{\partial q_k}{\partial x_k} - p \frac{\partial v_k}{\partial x_k} + \Phi, \quad q_k = -\lambda \frac{\partial T}{\partial x_k}, \tag{4}$$

$$f(p, \rho, T) = 0, \tag{5}$$

where Φ is a dissipative function, while the tensor components σ_{kj} are found from (1).

Let us examine the manner in which small perturbations in temperature, velocity, and density are propagated against the background of a volume of Maxwell fluid (mechanically in equilibrium, with a uniform stressed state) in the presence of a constant temperature gradient in the direction of the gravitational field. We will restrict ourselves to an examination of the perturbations propagating in a direction perpendicular to the gravitational field, in the form of steady-state plane waves. The entire set of temperature, viscosity (oscillations in the transverse component of velocity), and sonic waves propagating in the nonisothermal fluid in a gravitational field exclusively as a consequence of thermal compressibility will be referred to as thermal convection waves. All of the quantities in a state of mechanical equilibrium will be noted by the index zero, while small deviations from these quantities will be indicated by a prime. Then, following the familiar method of [5], we will seek the solution of system (1)-(5) in the form

$$v_k = 0 + v'_k(x, t), \quad \sigma_{kj} = \sigma^{(0)}_{kj} + \sigma'_{kj}(x, t), \quad \rho = \rho_0 + \rho'(x, t),$$
$$p = p_0 + p'(x, t), \quad T = T_0 + T'(x, t). \tag{6}$$

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Let us write the equations for the perturbations (we drop the primes):

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x}, \quad u = v_x, \tag{7}$$

$$\rho_0 \frac{\partial v}{\partial t} = \frac{\partial \sigma_{xy}}{\partial x} - \rho g_y, \quad v = v_y, \tag{8}$$

$$\tau \frac{\partial \sigma_{xx}}{\partial t} + \sigma_{xx} = \left(\frac{4}{3}\mu_0 + \eta_0\right)\frac{\partial u}{\partial x}, \tag{9}$$

$$\tau \frac{\partial \sigma_{xy}}{\partial t} + \sigma_{xy} = \mu_0 \frac{\partial v}{\partial x}, \tag{10}$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0, \tag{11}$$

$$\rho_0 c_0 \left(\frac{\partial T}{\partial t} + \gamma_1 v\right) = \lambda_0 \frac{\partial^2 T}{\partial x^2} - p_0 \frac{\partial u}{\partial x}, \tag{12}$$

$$\rho = \beta_1 T. \tag{13}$$

We assumed in the derivation of system (7)-(13) that the amplitude of the transverse oscillations of the fluid is considerably smaller than the wavelength, and we applied the same assumption with respect to the characteristic distance for the change in temperature, density, and pressure in mechanical equilibrium. The viscosity coefficients μ_0 and η_0, the coefficient of thermal conductivity λ_0, the heat capacity c_0, density ρ_0, and the pressure p_0 are therefore assumed to be constant. In the linearized equation of state the change in density with pressure in treated as a magnitude of the second order of smallness.

The equations for the temperature (12), for the transverse velocity component (8), for the stress-tensor component σ_{xy} (10), and for the density (13) are not associated with the remaining equations of the system and can be solved independently. Knowing ρ, from the continuity equation it is not difficult to find the longitudinal velocity component; then, from Eq. (9) we can find the stress-tensor component, and the pressure can be found from (7).

We will seek the solutions for (8), (11), (13), and (14) [sic] in the form

$$\begin{aligned} v &= V \exp i(\omega t - kx), \\ \sigma_{xy} &= \Pi \exp i(\omega t - kx), \\ T &= \theta \exp i(\omega t - kx). \end{aligned} \tag{14}$$

Having substituted (14) into (8), (11), (13), and (14) [sic], we will have

$$\begin{gathered} i\omega\rho_0 V + ik\Pi + \beta_1 g\theta = 0. \\ (i\omega\alpha + k^2\varkappa_1)\theta + \gamma_1 V = 0, \quad \alpha = 1 - \frac{p_0\beta_1}{\rho_0^2 c_0}, \quad \varkappa_1 = \frac{\lambda_0}{\rho_0 c_0}, \\ (1 + i\omega\tau)\Pi + ik\mu V = 0. \end{gathered} \tag{15}$$

A nontrivial solution for (15) is possible if the determinant of the system is different from zero. This condition enables us to write the dispersion equation which determines the relationship between the wave number and the frequency, i.e.,

$$\begin{gathered} (i\omega - \omega^2\tau + k^2\nu)(i\omega + k^2\varkappa) - \beta\gamma g(1 + i\omega\tau) = 0, \\ \beta_1 = \rho_0\beta, \quad \varkappa_1 = \alpha\varkappa, \quad \gamma_1 = \alpha\gamma. \end{gathered} \tag{16}$$

We will assume the wave number to be complex, i.e., $k = k_1 + ik_2$, and we will assume the frequency to be real. Separating the real and imaginary parts in (16) we derive two equations for the determination of the functions $\mathrm{Re}[k(\omega)] = k_1(\omega)$ and $\mathrm{Im}[k(\omega)] = k_2(\omega)$:

$$\begin{gathered} \omega^2 + \beta\gamma g + (k_1^2 - k_2^2)\omega^2\tau\varkappa + 2k_1k_2(\varkappa + \nu)\omega - [(k_1^2 - k_2^2)^2 - 4k_1^2k_2^2]\varkappa\nu = 0, \\ \omega^3\tau + \beta\gamma g\omega\tau - (k_1^2 - k_2^2)(\varkappa + \nu)\omega + 2k_1k_2\omega^2\tau\varkappa - 4k_1k_2(k_1^2 - k_2^2)\varkappa\nu = 0. \end{gathered} \tag{17}$$

The real part of the wave number k_1 is associated with the wavelength l by the relationship $l = 2\pi/k_1$. The imaginary part of the wave number k_2 determines the depth of penetration $L = k_2^{-1}$.

The depth of penetration is understood to refer to the distance at which the amplitude of the wave diminishes by a factor of e.

We begin the analysis of (17) with the case in which the waves are propagated without attenuation, i.e., $k_2 \equiv 0$. In a viscous heat-conducting medium this is possible only with synchronous compensation of the dissipated energy from external sources. In principle, this is possible in the case under consideration, since the required energy can be drawn from the constant gravitational and thermal fields. The solution of these equations shows that only waves of completely defined frequency and length can be propagated without attenuation in a given fluid; these waves are governed by the gravitational field, the temperature gradient, and the pressure gradient:

$$\omega = [\beta\gamma g \nu \varkappa \tau^2 - (\varkappa + \nu)^2]^{1/2} (\varkappa\tau)^{-1}, \tag{18}$$

$$k_1 = (\beta\gamma g \varkappa \tau^2 - \varkappa - \nu)^{1/2} \varkappa^{-1} \tau^{-1/2}. \tag{19}$$

Let us write out the expression for the phase velocity found with the undamped wave:

$$u_{\text{phase}} = \frac{\omega}{k_1} = \left[\frac{\beta\gamma g \nu \varkappa \tau^2 - (\varkappa + \nu)^2}{\tau (\beta\gamma g \varkappa \tau^2 - \varkappa - \nu)} \right]^{1/2}. \tag{20}$$

From the requirement that (18) and (20) be real, we have the following limitations on the magnitude and sign of the product $\beta\gamma g\tau^2$:

$$\beta\gamma g \tau^2 \geqslant \frac{\varkappa + \nu}{\varkappa}, \tag{21}$$

$$\beta\gamma g \tau^2 \geqslant \frac{(\varkappa + \nu)^2}{\varkappa \nu}. \tag{22}$$

If $\alpha > 0$, since $(\varkappa + \nu)^2 \geq (\varkappa + \nu)\nu$, the first of the inequalities written out is a consequence of the second; when $\alpha < 0$ and $|\varkappa| > \nu$ the second inequality is a consequence of the first. When $\alpha > 0$ we see from (22) that in a medium with a positive coefficient of thermal expansion the temperature gradient must be antiparallel to the gravitational field (heating from above). In a medium with a negative coefficient of thermal expansion the temperature gradient is parallel to the gravitational field (heating from above). Inequalities (21) and (22)can also be treated from the standpoint of limitations on the characteristic of an elastic fluid – the relaxation time τ. It follows from (21) and (22) that for the specified thermal and viscous properties of the fluid, and with specified temperature and gravitational fields, undamped thermal convection waves can be propagated in fluids with a relaxation time greater than $[(\varkappa + \nu)/\beta\gamma g\varkappa]^{1/2}$ or $[(\varkappa + \nu)^2/\beta\gamma g\varkappa\nu]^{1/2}$. Consequently, the undamped thermal convection waves cannot propagate in purely viscous heat-conducting fluids. However, in such fluids, as will be demonstrated below, we can have the propagation of weakly damped thermal convection waves.

Let us present several numerical estimates. Let $\varkappa \sim \nu \sim 10^{-5}$ m^2/sec, $\beta g \sim 10^{-2}$ m/sec·deg, which corresponds to the characteristics of gases, whereas when $\gamma \sim 1$ deg/m we have $\tau \gtrsim 1$ sec, for $\gamma \sim 10^2$ deg/m we have $\tau \gtrsim 10^{-1}$ sec. Let $\nu \sim 10^{-2}$ m^2/sec, $\varkappa \sim 10^{-7}$ m^2/sec, $\beta g \sim 10^{-5}$ m/deg·sec, as, for example, in very viscous fluids, then for $\gamma \sim 1$ deg/m we have $\tau \geq 10^5$ sec, while for $\gamma \sim 10^2$ deg/m we have $\tau \gtrsim 10^4$ sec.

Let us present certain asymptotic estimates. With an increase in the relaxation time τ the frequency in (18) becomes an increasingly weaker function of τ and at the limit tends to $\omega = (\beta\gamma g\nu/\varkappa)^{1/2}$, and the square of the wave number as a function of the relaxation time, in this case, becomes linear, i.e., $k^2 \simeq \tau\beta\gamma g/\varkappa$. Thus at the limit of large τ, the length and velocity of the thermal convection waves diminish as $\tau^{-1/2}$.

2. Let us turn to an examination of the damped thermal convection waves. We are primarily interested in the weakly attenuated waves in purely viscous fluids where $\tau \equiv 0$. The existence of weakly attenuated waves in a Maxwell fluid with frequencies and wavelengths close to those examined in the previous section is regarded as self-evident. The solution of the dispersion equations (17) at the limit as $\tau \to 0$ presents no difficulty. Omitting the intervening calculations, we write the final results:

a) $\omega^2 (\nu - \varkappa)^2 - 4\pi\nu\beta\gamma g = r > 0.$

$$k_1^2 = k_2^2,$$

$$k_2 = -\left[\frac{\omega(\varkappa+\nu) \pm r^{1/2}}{4\varkappa\nu}\right]^{1/2}, \quad \beta\gamma > 0, \tag{23}$$

$$k_2 = -\left[\frac{\pm\omega(\varkappa+\nu) + r^{1/2}}{4\varkappa\nu}\right]^{1/2}, \quad \beta\gamma < 0, \quad |\beta\gamma| \gg \omega^2;$$

b) $r \leqslant 0$.

$$k_1 k_2 = -\frac{\omega(\varkappa+\nu)}{4\varkappa\nu} = -A, \tag{24}$$

$$k_2 = -\left[\mp B + \sqrt{B^2+A^2}\right]^{1/2}, \quad B = \frac{r^{1/2}}{4\varkappa\nu}. \tag{25}$$

We see from expressions (23)-(25) that depending on the absolute value and sign of $\beta\gamma g\omega^{-2}$, in principle, various thermal convection waves can be propagated in the fluid. For all frequencies ω under the condition that $\beta\gamma < 0$ and for frequencies $\omega \geq 2(\varkappa\nu\beta\gamma g)^{1/2}|\nu - \varkappa|^{-1}$ when $\beta\gamma > 0$ highly damped waves (24) propagate. Indeed, the depth of penetration in this case is of the same order of magnitude as the wavelength $L \sim l$. This process of propagating perturbations is essentially aperiodic and it is proper that it be referred to as an oscillatory process, rather than a wave process.

The situation is different with the propagation of perturbations for frequencies $\omega < 2(\varkappa\nu\beta\gamma g)^{1/2}|\nu - \varkappa|^{-1}$ in a liquid with $\beta\gamma > 0$. Here, as we can see from (24) and (25), two waves are propagated with the specified frequency. One, corresponding to the plus sign in front of the B in (25), is a strongly damped wave. The other, whose wave-vector components we write in the form

$$k_1 = A[B(\sqrt{1+A^2B^{-2}} - 1)]^{-1/2}, \quad k_2 = -[B(\sqrt{1+A^2B^{-2}} - 1)]^{1/2}, \tag{26}$$

may be weakly damped under certain conditions.

Let us determine these conditions. First of all, we must require a large absolute value for the depth of attenuation L. This is possible when $A^2B^{-2} \ll 1$. Secondly, the depth of penetration must be large in the scale of the wavelengths. In answer to the question as to when this is satisfied, we must analyze the expression L/l:

$$\frac{L}{l} = \frac{k_1}{2\pi k_2} = \frac{A}{2\pi B(\sqrt{1+A^2B^{-2}} - 1)}. \tag{27}$$

Considering the condition $A^2B^{-2} \ll 1$, we simplify expression (27) to

$$\frac{L}{l} \simeq \frac{[4\varkappa\nu\beta\gamma g - \omega^2(\nu-\varkappa)^2]^{1/2}}{\pi\omega(\varkappa+\nu)}.$$

It is not difficult to see that the ratio L/l will be the larger, the greater the coefficient of thermal expansion and the greater the temperature gradient, the smaller the frequency, and the closer the numerical values of the coefficients of kinetic viscosity and thermal diffusivity. The absolute magnitude of the temperature gradient is bounded from above by the condition that mechanical equilibrium prevail, so that there is a finite upper limit for the frequencies which are propagated with low attenuation, i.e., $\omega < (\beta\gamma g)^{1/2}$. As demonstrated by calculations, the low-frequency oscillations are weakly attenuated. We will present the numerical estimates. Best from the standpoint of high-frequency transmission are media with $\varkappa \simeq \nu$, e.g., air $p \sim 1$ atm, $T \sim 15\,°C$, $\nu \sim 10^{-6}$ m^2/sec, and $\varkappa \sim 2 \cdot 10^{-6}$ m^2/sec. With a temperature gradient of 10^2 deg/m at a frequency of 10^{-3} Hz the depth of penetration is ~10 m or ~10^3 wavelengths, and with a frequency of 10^{-4} Hz we have $L \sim 10^2$ m and $L/l \sim 10^4$.

Weakly damped thermal convection waves of specific frequencies and lengths can thus be propagated in nonisothermal viscous liquids with a temperature gradient parallel (antiparallel) to the gravitational field in a direction perpendicular to $\vec{g}$ and ΔT. The greatest interest with regard to thermal convection waves is apparently centered on the possibility the propagation of weakly damped transverse and thermal waves, since in an isothermal liquid in mechanical equilibrium (without consideration of the forces of gravity) the temperature and viscous waves are strongly damped [5, 6].

The thermal convection waves predicted and studied in this paper may be used to develop a heat source of transverse oscillations, as well as for the measurement of elastic, viscous, and thermal properties of a liquid, the design of a mechanical heat-oscillation generator, and for similar purposes.

LITERATURE CITED

1. A. S. Lodge, Elastic Fluids [Russian translation], Nauka, Moscow (1968).
2. W. L. Wilkinson, Non-Newtonian Fluids [Russian translation], Mir, Moscow (1964).
3. J. C. Maxwell, Phil. Mag., 35, 129-137 (1868).
4. A. V. Luikov and B. M. Berkovskii, in: Heat and Mass Transfer in Non-Newtonian Fluids [in Russian], Énergiya, Moscow (1968).
5. L. D. Landau and E. M. Livshits, The Mechanics of Continuous Media [in Russian], GITTL, Moscow (1953).
6. A. V. Luikov and Yu. A. Mikhailov, The Theory of Heat and Mass Transfer [in Russian], Gosénergoizdat (1963).